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Vibrations of Pressurized Orthotropic Cylindrical Membranes

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This paper presents the analysis of the free and forced vibrations of pressurized orthotropic cylindrical membrane shells. The following points are brought out. 1) The response of the shell is very highly dependent on the internal pressure, and on the relative magnitude of the applied pressure loading as compared to the internal pressure. In fact, it is clearly demonstrated that when the two pressures are of the same order of magnitude, a linearized analysis is not sufficient to discuss the complete behavior. 2) The behavior of the shell is also very significantly affected by the values of the various elastic constants. The ratio of the circumferential stiffness to the axial stiffness was found to be a particularly important parameter, with somewhat less importance being attached to the relative magnitude of the inplane shear modulus. 3) Finally, the simplification of the analysis by the deletion of the inplane inertia terms makes little difference in the results of the forced vibration analysis. This is a particularly important simplification to recognize if a full nonlinear analysis is to be carried out.

Nomenclature

b,d,α_1	= dimensions defined in Fig. 8
C_{ij},G_{12}	= orthotropic material constants
h	= thickness of cylinders
K	= dimensionless frequency parameter
L	= length of cylinder
n,λ	= circumferential and axial wave numbers
p	= initial internal pressure
p_x,p_y,p_z	= applied surface tractions
q_{mn}	= mode participation factor
R	= radius of cylinder
t_x,t_y,t_{xy}	= dimensionless stress components
t_{x0},t_{y0},t_{xy0}	= initial stress state
$\tilde{t}_x,\tilde{t}_y,\tilde{t}_{xy}$	= stress perturbations
U_{mn},V_{mn},W_{mn}	= modal displacements
u,v,w	= dimensionless displacements
u^*,v^*,w^*	= axial, circumferential, and inward normal displacements
u_0,v_0,w_0	= initial displacements
$\tilde{u},\tilde{v},\tilde{w}$	= displacement perturbations
x,y	= axial and circumferential coordinates
α,β,γ	= dimensionless material constants
$\epsilon_x,\epsilon_y,\gamma_{xy}$	= strain components
ξ,φ	= dimensionless coordinates
ρ	= mass density
τ	= dimensionless time

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Introduction

THIS work presents a study of the dynamic response of a pressurized orthotropic cylindrical membrane in both free and forced vibration configurations. The effects of internal pressurization, inplane inertia terms, and variations in the elastic constants will be examined.

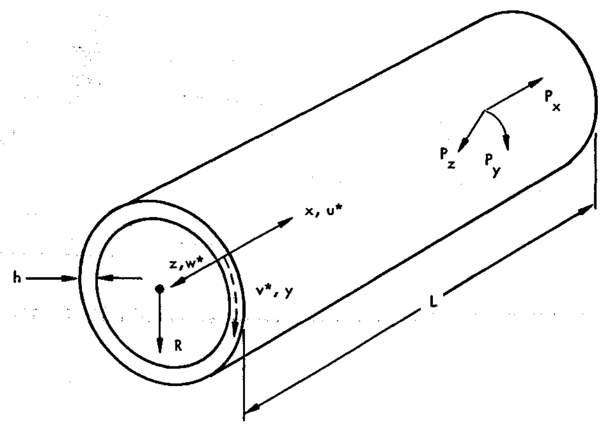

Fig. 1 Geometry.

The behavior of extremely thin initially stressed shells is such that classical methods are inadequate for their proper analysis. In recent years considerable effort has been devoted to the formation of consistent shell theories,[1–3] and to the development of solutions for nonlinear membrane shell problems.[4–6] Another area of research has been the representation of nonlinear problems as a sequence of linear problems superposed on already known linear or nonlinear solutions. Such work has been done for both prestressed membrane shells,[7–10] and prestressed shells in which bending effects are important.[4,11,12] Some analyses of the free vibrations of various thin shells of revolution with initial stress have been reported.[4,7,8,11–15] Far less work has been reported for forced vibrations of shells (see, for instance, Ref. 16) and it appears that none of this work has included the effects of orthotropy, internal pressurization, or the evaluation of the effects of the inplane inertia terms.

The effect of pressurization has been discussed extensively for isotropic membranes by Reissner[7] who pointed out that the frequency of free vibration of the radial mode is raised by internal pressure. Similar results are reported by Leonard[10] and by Liepins.[15] Some of the effects of variations in the elastic constants of thick shells are discussed by Klosner and Dym.[17] Finally, the effects of inplane inertia are discussed by Reissner[4,13] and by Forsberg,[18] who found that the natural frequencies of the radial mode could be significantly higher if the tangential inertia were deleted.

Kalnins has pointed out that deletion of inplane inertia terms yields a loss of membrane modes, and that the bending modes for a nonshallow shell predicted by membrane theory can be in error.[19,20] The first of these phenomena seems expected, and is found in this investigation. In regard to the second observation, it should hardly play any role at all in an investigation based on shallow shell theory.

The present work is concerned with determining not only the effects of pressurization, and of deletion of the inplane inertia, but also the effects of variations of the elastic constants on the free and forced motions of cylindrical membrane shells. The equations used are based on the nonlinear shell equations commonly used for cylinders, modified to delete the bending terms and to include orthotropic elastic constants. Solutions are obtained in terms of the normal modes of free vibrations, which are taken here to be of the Rayleigh type. It will be shown that variations in the elastic constants and in the internal pressure can have a significant effect on the behavior of the shell.

Derivation of Equations

The equations that govern the forced vibrations of an orthotropic pressurized membrane subjected to loading that may vary both spatially and in time are derived herein. To allow for reasonably large deformations, the formulation will consider the nonlinear strain displacement relations used as midsurface kinematic equations in shallow shell theory.

Consider a circular cylinder of a wall thickness h, length L, and radius R. The axial, circumferential, and inward radial displacement components are u^*, v^*, and w^*, respectively, whereas x and y denote axial and circumferential coordinates; see Fig. 1. Because a membrane is "thin" it may be analyzed as a two-dimensional body. The appropriate strain-displacement equations are[12]:

$$\begin{aligned} \epsilon_x &= (R/L)\partial u/\partial\xi + \tfrac{1}{2}(R/L)^2(\partial w/\partial\xi)^2 \\ \epsilon_y &= \partial v/\partial\varphi - w + \tfrac{1}{2}(\partial w/\partial\varphi)^2 \\ \gamma_{xy} &= \partial u/\partial\varphi + (R/L)\partial v/\partial\xi + (R/L)(\partial w/\partial\xi)\partial w/\partial\varphi \end{aligned} \tag{1}$$

where the following dimensionless quantities have been introduced:

$$\begin{gathered} u = u^*/R,\ v = v^*/R,\ w = w^*/R \\ \xi = x/L,\ \varphi = y/R \end{gathered} \tag{2}$$

The two-dimensional linear orthotropic constitutive relations for a conservative material are[17]

$$\begin{aligned} t_x &= \epsilon_x + \alpha\epsilon_y \\ t_y &= \alpha\epsilon_x + \beta\epsilon_y \\ t_{xy} &= \gamma\gamma_{xy} \end{aligned} \tag{3}$$

where the t_{ij} are stress components divided by the longitudinal elastic constant C_{11}, and α, β, γ are dimensionless material parameters given by

$$\alpha = C_{12}/C_{11},\ \beta = C_{22}/C_{11},\ \gamma = G_{12}/C_{11} \tag{4}$$

Having the appropriate constitutive relations, Eqs. (3) and kinematic conditions, Eqs. (1), Hamilton's principle can be utilized to formulate the governing equations of equilibrium. The Hamilton principle states that during a virtual variation of the displacements of a system between two instants of time, t_0 and t_1, the Lagrangian L attains an extreme value if its first variation vanishes;

$$\delta\int_{t_0}^{t_1} L\,dt = 0 \tag{5}$$

For an elastic system, the Lagrangian takes the form

$$L = T - (U_e + V_p) \tag{6}$$

where T is the kinetic energy, U_e the strain energy, and V_p is the potential of the applied loading. Introducing a dimensionless time τ such that

$$\tau = (C_{11}/\rho R^2)^{1/2}t \tag{7}$$

the kinetic energy in the shell is

$$T = \frac{C_{11}hRL}{2}\int_0^1\int_0^{2\pi}\left[\left(\frac{\partial u}{\partial\tau}\right)^2 + \left(\frac{\partial v}{\partial\tau}\right)^2 + \left(\frac{\partial w}{\partial\tau}\right)^2\right]d\xi d\varphi \tag{8}$$

The strain energy U_e may be written in the form

$$U_e = \frac{C_{11}hRL}{2}\int_0^1\int_0^{2\pi}[t_x\epsilon_x + t_y\epsilon_y + t_{xy}\gamma_{xy}]d\xi d\varphi \tag{9}$$

Finally, if a uniform internal pressure p, and applied surface tractions p_x, p_y, p_z acting in the axial, circumferential, and radial directions are considered, the potential of the applied loads is

$$V_p = C_{11}hRL\int_0^1\int_0^{2\pi}\left[\frac{p_xR}{C_{11}h}u + \frac{p_yR}{C_{11}h}v + \frac{(p_z - p)R}{C_{11}h}w\right]d\xi d\varphi \tag{10}$$

Substituting Eqs. (6–10) into the statement of Hamilton's principle, Eq. (5), and carrying out the indicated variation, yields the governing differential equations and appropriate boundary conditions for finite deformations of a vibrating orthotropic cylindrical membrane. When written in terms of the dimensionless stresses, the differential equations take the form

$$\left(\frac{R}{L}\right)\frac{\partial t_x}{\partial \xi} + \frac{\partial t_{xy}}{\partial \varphi} = \frac{\partial^2 u}{\partial \tau^2} - \frac{p_x R}{C_{11}h}$$

$$\left(\frac{R}{L}\right)\frac{\partial t_{xy}}{\partial \xi} + \frac{\partial t_y}{\partial \varphi} = \frac{\partial^2 v}{\partial \tau^2} - \frac{p_y R}{C_{11}h}$$

$$\left(\frac{R}{L}\right)^2 \frac{\partial}{\partial \xi}\left(t_x \frac{\partial w}{\partial \xi}\right) + \frac{\partial}{\partial \varphi}\left(t_y \frac{\partial w}{\partial \varphi}\right) + \left(\frac{R}{L}\right)\frac{\partial}{\partial \xi}\left(t_{xy} \frac{\partial w}{\partial \varphi}\right) + \left(\frac{R}{L}\right)\frac{\partial}{\partial \varphi}\left(t_{xy} \frac{\partial w}{\partial \xi}\right) + t_y = \frac{\partial^2 w}{\partial \tau^2} + \frac{pR}{C_{11}h} - \frac{p_z R}{C_{11}h} \tag{11}$$

The boundary conditions at the ends of the shell ($\xi = 0, 1;\ x = 0, L$) are then

$$\begin{aligned} &\text{either } t_{xy} = 0 \text{ or } v \text{ prescribed} \\ &\text{either } t_x = 0 \text{ or } u \text{ prescribed} \\ &\text{either } \left(\frac{R}{L}\right) t_x \frac{\partial w}{\partial \xi} + t_{xy}\frac{\partial w}{\partial \varphi} = 0 \text{ or } w \text{ prescribed} \end{aligned} \tag{12}$$

If the shell was not complete in the circumferential direction so that the dimensionless coordinate was restricted as $\varphi_0 \leq \varphi \leq \varphi_1$ (rather than $0 \leq \varphi \leq 2\pi$ for the complete shell), the appropriate conditions at these edges would be found to be

$$\begin{aligned} &\text{either } t_{xy} = 0 \text{ or } u \text{ prescribed} \\ &\text{either } t_y = 0 \text{ or } v \text{ prescribed} \\ &\text{either } \left(\frac{R}{L}\right) t_{xy} \frac{\partial w}{\partial \xi} + t_y\frac{\partial w}{\partial \varphi} = 0 \text{ or } w \text{ prescribed} \end{aligned} \tag{13}$$

For a complete circular shell, those conditions are replaced by continuity conditions, i.e.,

$$u(\varphi) = u(\varphi + 2\pi), \text{ etc.} \tag{14}$$

It is interesting to note that in the linear theory of membrane shells, the third boundary condition of Eqs. (12) and (13) does not arise. Only in-plane conditions are permissible in the linear theory.

It is evident that Eqs. (11) are highly nonlinear and that solutions will be difficult to obtain. Disregarding the inplane inertia terms in the axial and circumferential directions, it would be possible to introduce a stress function that would identically satisfy the in-plane equilibrium equation—the first two equations of (11). It would then be possible to reduce the system (11) to a pair of coupled equations with only two unknowns, the stress function and the radial displacement. Such a procedure would parallel, of course, the well-known von Kármán theory of plates, and the Marguerre and von Kármán-Donnell theories of shallow shells.

Since the present analysis will be linearized, the two inertia terms as discussed previously will not be discarded. Rather, a linearized analysis to assess the importance of these terms shall be attempted. If the gross effect of these terms can be demonstrated to be small, the analysis of the full nonlinear problem may be considerably simplified.

Linearization

The nonlinear partial differential Eqs. (11) may be linearized to simplify their analysis. As a basic state of stress, tonsider the simple static deformation of a cylinder subjected aco uniform internal pressure. The problem is statically determinate, and for closed ends the simple stress state may be obtained:

$$t_{x0} = pR/2C_{11}h,\ t_{y0} = pR/C_{11}h,\ t_{xy0} = 0 \tag{15}$$

If the nonlinear terms in the kinematic relations in Eqs. (1) are deleted, they can be used in conjunction with Eqs. (3) to obtain the displacements for the uniform pressure state;

$$u_0 = \frac{pR}{2C_{11}h}\left(\frac{L}{R}\right)\left(\frac{\beta - 2\alpha}{\beta - \alpha^2}\right)\xi,\ v_0 = 0,\ w_0 = -\frac{pR}{2C_{11}h}\left(\frac{2 - \alpha}{\beta - \alpha^2}\right) \tag{16}$$

Of course the solutions (16) assume a cylinder that is uniformly expanding. If the ends were inextensible, then Eqs. (15) and (16) would not apply.

To linearize the partial differential equations (11), it can be assumed that the total stresses (t_x, t_{xy}, t_y) and the displacements (u, v, w) are made up of the linear, uniform membrane stresses and displacements of Eqs. (15) and (16), and small excursions from this static equilibrium state. Thus

$$\begin{aligned} t_x = t_{x0} + \tilde{t}_x, \quad t_y = t_{y0} + \tilde{t}_y, \quad t_{xy} = t_{xy0} + \tilde{t}_{xy} \\ u = u_0 + \tilde{u}, \quad v = v_0 + \tilde{v}, \quad w = w_0 + \tilde{w} \end{aligned} \tag{17}$$

where the perturbation quantities $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{t}_x$, $\tilde{t}_{xy}$, and $\tilde{t}_y$ are small enough so that their squares and products are negligible when the forms (17) are substituted into Eqs. (11). The following equations then relate to the perturbation quantities:

$$\begin{aligned} &\left(\frac{R}{L}\right)\frac{\partial \tilde{t}_x}{\partial \xi} + \frac{\partial \tilde{t}_{xy}}{\partial \varphi} = \frac{\partial^2 \tilde{u}}{\partial \tau^2} - \frac{p_x R}{C_{11}h} \\ &\left(\frac{R}{L}\right)\frac{\partial \tilde{t}_{xy}}{\partial \xi} + \frac{\partial \tilde{t}_y}{\partial \varphi} = \frac{\partial^2 \tilde{v}}{\partial \tau^2} - \frac{p_y R}{C_{11}h} \\ &\left(\frac{R}{L}\right)^2 t_{x0}\frac{\partial^2 \tilde{w}}{\partial \xi^2} + t_{y0}\frac{\partial^2 \tilde{w}}{\partial \varphi^2} + \tilde{t}_y = \frac{\partial^2 \tilde{w}}{\partial \tau^2} - \frac{p_z R}{C_{11}h} \end{aligned} \tag{18}$$

The linearized equilibrium equations (18) can also be written entirely in terms of the perturbation displacements;

$$\left(\frac{R}{L}\right)^2 \frac{\partial^2 \tilde{u}}{\partial \xi^2} + \gamma\frac{\partial^2 \tilde{u}}{\partial \varphi^2} + (\alpha + \gamma)\left(\frac{R}{L}\right)\frac{\partial^2 \tilde{v}}{\partial \xi \partial \varphi} - \alpha\left(\frac{R}{L}\right)\frac{\partial \tilde{w}}{\partial \xi} = \frac{\partial^2 \tilde{u}}{\partial \tau^2} - \frac{p_x R}{C_{11}h}$$

$$(\alpha + \gamma)\frac{R}{L}\frac{\partial^2 \tilde{u}}{\partial \xi \partial \varphi} + \gamma\left(\frac{R}{L}\right)^2\frac{\partial^2 \tilde{v}}{\partial \xi^2} + \beta\frac{\partial^2 \tilde{v}}{\partial \varphi^2} - \beta\frac{\partial \tilde{w}}{\partial \varphi} = \frac{\partial^2 \tilde{v}}{\partial \tau^2} - \frac{p_y R}{C_{11}h} \tag{19}$$

$$\frac{pR}{2C_{11}h}\left(\frac{R}{L}\right)^2\frac{\partial^2 \tilde{w}}{\partial \xi^2} + \frac{pR}{C_{11}h}\frac{\partial^2 \tilde{w}}{\partial \varphi^2} + \alpha\left(\frac{R}{L}\right)\frac{\partial \tilde{u}}{\partial \xi} + \beta\frac{\partial \tilde{v}}{\partial \varphi} - \beta\tilde{w} = \frac{\partial^2 \tilde{w}}{\partial \tau^2} - \frac{p_z R}{C_{11}h}$$

It is of interest to mention that isotropic counterparts to Eqs. (19) may be derived from a very different point of view than that represented here. The equations of motion for an initially stressed membrane of arbitrary meridional contour have been derived elsewhere[9,10] using the theory of incremental deformations.[21] In these more accurate equations if the effect of the initial stress state is neglected in comparison to unity, then the two derivations yield the same result.[22]

Free Vibration Solution

To examine the free vibration frequencies of a pressurized orthotropic membrane shell, a set of solutions to the homoge-

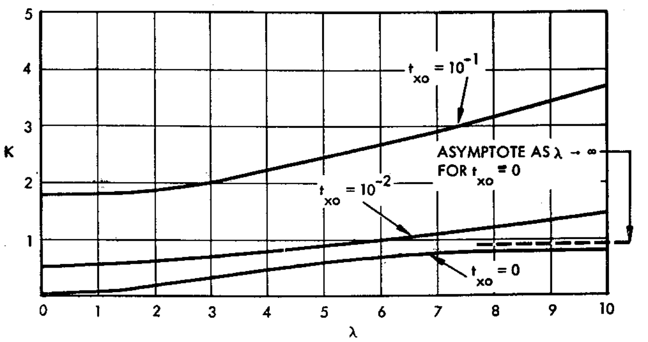

Fig. 2 Frequency factor vs axial wavelength factor: no tangential inertia, isotropic, $n = 4$.

neous versions of Eqs. (19) are taken in the form

$$\begin{aligned} \tilde{u}(\xi,\varphi,\tau) &= A_{mn} \cos m\pi\xi \cos n\varphi \cos K\tau \\ \tilde{v}(\xi,\varphi,\tau) &= B_{mn} \sin m\pi\xi \sin n\varphi \cos K\tau \\ \tilde{w}(\xi,\varphi,\tau) &= C_{mn} \sin m\pi\xi \cos n\varphi \cos K\tau \end{aligned} \tag{20}$$

where

$$K = (\rho/C_{11})^{1/2}R\omega = \text{frequency factor} \tag{21}$$

where ω is the frequency of the system expressed in rad/sec.

The Rayleigh type solution (20) satisfies the boundary conditions;

$$t_x = 0, v = w = 0 \text{ at } \xi = 0, 1 \tag{22}$$

These correspond to supporting the shell at its ends with a ring that is rigid in its own plane, but unable to support forces in the direction normal to that plane.

Substitution of the solutions (20) into the differential equations (19) yields a system of three linear algebraic equations for the coefficients A_{mn}, B_{mn}, C_{mn}. Written in matrix form,

$$\begin{bmatrix} (K^2 - \lambda^2 - n^2\gamma) & (\alpha + \gamma)n & -\alpha\lambda \\ (\alpha + \gamma)n & (K^2 - \gamma\lambda^2 - n^2\beta) & \beta n \\ -\alpha\lambda & \beta n & (K^2 - t_{x0}\lambda^2 - t_{y0}n^2 - \beta) \end{bmatrix} \begin{Bmatrix} A_{mn} \\ B_{mn} \\ C_{mn} \end{Bmatrix} = 0 \tag{23}$$

In (23) the axial-wavelength parameter $\lambda = m\pi R/L$ has been introduced for simplicity. A nontrivial solution to the system (23) can exist only if the determinant of the coefficients vanishes. This yields a cubic equation in the square of the frequency factor K. Thus for each mode (each pair m, n, or each pair λ, n) three frequencies are found. The lowest one corresponds to predominantly radial motion, and the higher two are linked with predominantly tangential modes of vibration.

It is possible to obtain a much simpler solution for the lowest (radial-mode) frequency. This is accomplished by deleting the inplane inertia terms $\partial^2 u/\partial\tau^2$, $\partial^2 v/\partial\tau^2$ in Eqs. (19) before substituting the solutions (20). If this is done it is a straightforward operation to find the following explicit

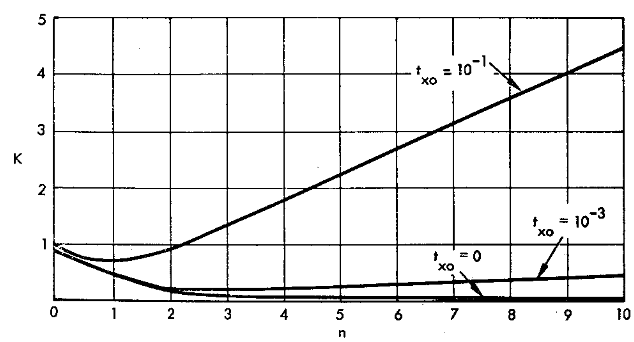

Fig. 3 Frequency factor vs circumferential wave number: no tangential inertia, isotropic, $\lambda = 1$.

Table 1 Effect of in-plane inertia and initial stress on lowest frequency factor (isotropic: $\alpha = 0.30$, $\beta = 1.00$, $\gamma = 0.35$, $\lambda = 1.0$)

n	K (with in-plane inertia) $t_{x0} = 0$	K (no in-plane inertia) $t_{x0} = 0$	K (with in-plane inertia) $t_{x0} = 10^{-1}$	K (no in-plane inertia) $t_{x0} = 10^{-1}$
1	0.3523	0.4770	0.5090	0.7263
2	0.1687	0.1908	0.8237	0.9677
3	0.0904	0.0954	1.2770	1.3823
4	0.0543	0.0561	1.7390	1.8172
5	0.0359	0.0367	2.1972	2.2589
6	0.0254	0.0258	2.6522	2.7022
7	0.0189	0.0191	3.1040	3.1461
8	0.0146	0.0147	3.5553	3.5918
9	0.0116	0.0116	4.0051	4.0371
10	0.0095	0.0094	4.4539	4.4833

equation for the radial frequency factor:

$$K^2 = t_{x0}\lambda^2 + t_{y0}n^2 + \frac{(\beta - \alpha^2)\lambda^4}{\lambda^4 + [(\beta - \alpha)/\gamma - 2\alpha]n^2\lambda^2 + \beta n^4} \tag{24}$$

For the isotropic case, Eq. (24) reduces to

$$K^2 = t_{x0}\lambda^2 + t_{y0}n^2 + (1 - \nu^2)\lambda^4/(\lambda^2 + n^2)^2 \tag{25}$$

The result (25) is given by Reissner[7] and is obtainable from the results of Fung et al.,[11] by elimination of the appropriate bending contributions.

The results obtained from numerical calculations through the vanishing of the determinant of Eq. (23) and of the explicit formula of Eq. (24) are presented in Tables 1–3 and Figs. 2–7. It is evident from the tables that as n and λ increase, the frequency determined by deleting the tangential inertia agrees quite well with the low frequency of the triplet of frequencies which are obtained when all the inertia terms are retained. Note that this is true for a significant range of values of the pressurization. A comparison of Tables 1 and 2 indicates that a decrease in the parameter β, the ratio of the circumferential extensional stiffness to the axial stiffness, yields a decrease in the natural frequency. However, when the pressurization is not zero, as n or λ become very large, the effect of variation in the elastic constants becomes negligible. This is evident not only from the tables, but from Eqs. (24) and (25).

Figures 2–4 give a clearer picture of the pressurization effect. In the absence of pressurization, from Eq. (24), it is seen that as $\lambda \to 0$, $K^2 \to 0$, and as $n \to 0$, $K^2 \to (\beta - \alpha^2)$. Thus definite asymptotes exist when there is no internal pressure. If an initial prestress exists, then as n or λ become indefinitely large, the frequency becomes indefinitely large. This is clearly indicated in Figs. 2 and 3. Figure 4 indicates

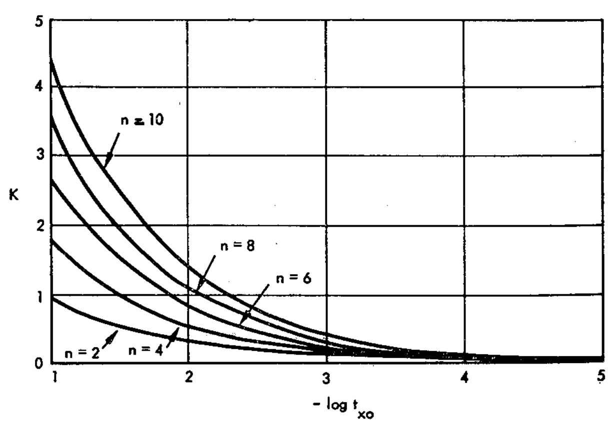

Fig. 4 Frequency factor vs log (pressurization factor): no tangential inertia, isotropic, $\lambda = 1$.

Table 2 Effect of in-plane inertia and initial stress on lowest frequency factor (orthotropic: $\alpha = 0.30$, $\beta = 0.20$, $\gamma = 0.30$, $\lambda = 1.0$)

n	K (with in-plane inertia) $t_{x0} = 0$	K (no in-plane inertia) $t_{x0} = 0$	K (with in-plane inertia) $t_{x0} = 10^{-1}$	K (no in-plane inertia) $t_{x0} = 10^{-1}$
1	0.3162	0.3373	0.4169	0.6433
2	0.1592	0.1835	0.5369	0.9663
3	0.0796	0.0854	0.9126	1.3814
4	0.0459	0.0476	1.3491	1.8171
5	0.0296	0.0302	1.8022	2.2578
6	0.0206	0.0209	2.2620	2.7020
7	0.0152	0.0153	2.7230	3.1463
8	0.0116	0.0117	3.1788	3.5923
9	0.0092	0.0092	3.6377	4.0371
10	0.0074	0.0074	4.0955	4.4830

rather dramatically the effect of pressurization on the frequency of the radial mode.

Figures 5–7 were obtained from the free vibration solution of the full problem; that is, all the inertia terms are retained. The effect of varying the elastic constants is clearly indicated in all the figures. The lowest frequency is most affected by a drop in β and somewhat less by a drop in γ, the ratio of the inplane shear modulus to the axial extensional rigidity. Figure 6 indicates that the middle frequency (axial mode) is less sensitive to variations in the elastic constants, and still less to variations of the internal pressure. As might be expected, the highest frequency (circumferential) mode is almost completely insensitive to variations in the elastic constants or the internal pressure, as is seen in Fig. 7.

Forced Vibration Solution

To obtain solutions to the forced vibration problem represented by the partial differential equations (19), the boundary conditions (22), and initial conditions that are as yet unspecified, it is assumed that the displacements can be expressed in the form

$$\begin{aligned} \tilde{u}(\xi,\varphi,\tau) &= \Sigma\Sigma\, U_{mn}(\xi,\varphi)q_{mn}(\tau) \\ \tilde{v}(\xi,\varphi,\tau) &= \Sigma\Sigma\, V_{mn}(\xi,\varphi)q_{mn}(\tau) \\ \tilde{w}(\xi,\varphi,\tau) &= \Sigma\Sigma\, W_{mn}(\xi,\varphi)q_{mn}(\tau) \end{aligned} \tag{26}$$

where $q_{mn}(\tau)$ is the mode participation factor and where the spatial components are the mode shapes of the free vibrations. The procedure by which the mode participation factors are obtained is straightforward. (For details see Ref. 16.) Suffice it to say here that the coordinates are obtained as solutions of the ordinary differential equation

$$d^2q_{mn}/d\tau^2 + K_{mn}{}^2q_{mn} = Q_{mn}(\tau) \tag{27}$$

Table 3 Effect of in-plane inertia and initial stress on lowest frequency factor (orthotropic: $\alpha = 0.30$, $\beta = 0.20$, $\gamma = 0.30$, $n = 4$)

λ	K (with in-plane inertia) $t_{x0} = 0$	K (no in-plane inertia) $t_{x0} = 0$	K (with in-plane inertia) $t_{x0} = 10^{-1}$	K (no in-plane inertia) $t_{x0} = 10^{-1}$
1	0.0459	0.0476	1.3491	1.8172
2	0.1767	0.1835	1.1273	1.9060
3	0.2969	0.3006	1.2720	2.0473
4	0.3364	0.3373	1.6888	2.2171
5	0.3430	0.3434	2.2160	2.4124
6	0.3425	0.3428	2.6272	2.6300
7	0.3409	0.3412	2.8641	2.8660
8	0.3393	0.3395	3.1155	2.1171
9	0.3380	0.3382	3.3774	3.3781
10	0.3370	0.3371	3.6473	3.6490

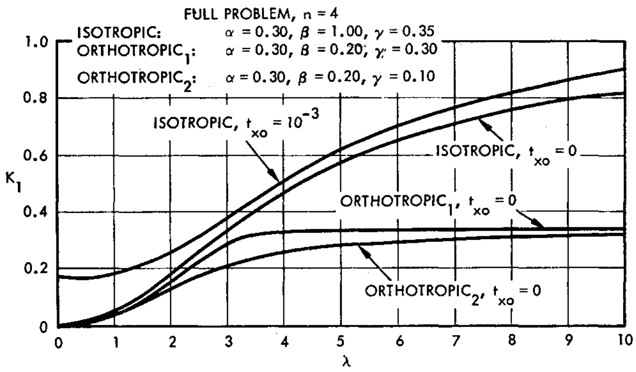

Fig. 5 Lowest frequency factor vs axial wavelength factor: full problem, $n = 4$.

where K_{mn} is the frequency factor, and

$$Q_{mn}(\tau) = \int_0^1 \int_0^{2\pi} \left[\frac{p_xR}{C_{11}h}\,U_{mn} + \frac{p_yR}{C_{11}h}\,V_{mn} + \frac{p_zR}{C_{11}h}\,W_{mn}\right]d\varphi d\xi \tag{28}$$

Note that use has been made of the normalizing condition

$$\int_0^1 \int_0^{2\pi} (U_{mn}{}^2 + V_{mn}{}^2 + W_{mn}{}^2)d\varphi d\xi = 1 \tag{29}$$

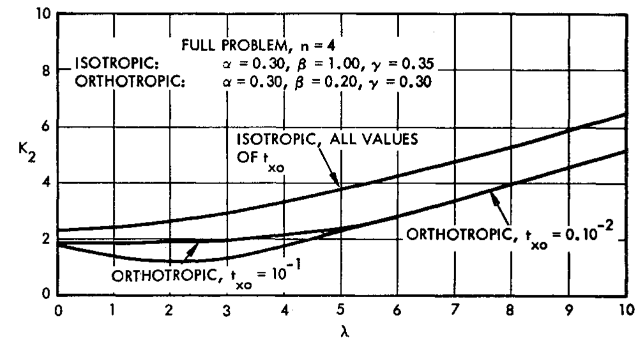

Fig. 6 Middle frequency factor vs axial wavelength factor: full problem, $n = 4$.

If the shell is initially at rest, the general solution to (27) is given by

$$q_{mn}(\tau) = \frac{1}{K_{mn}} \int_0^\tau Q_{mn}(\tau') \sin[K_{mn}(\tau-\tau')]d\tau' \tag{30}$$

It might be noted here that when all the inertia terms are retained, there is a triplet of frequencies obtained for each

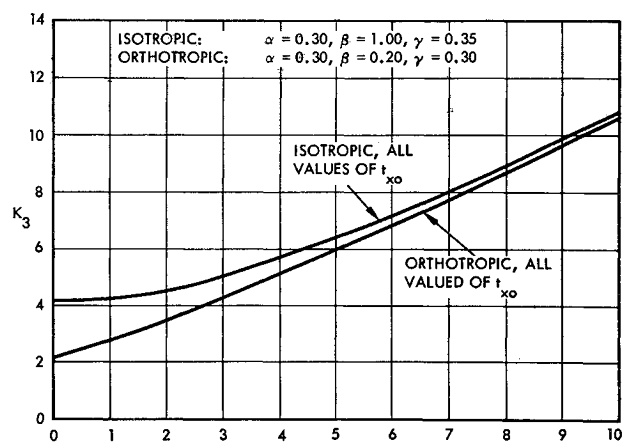

Fig. 7 Highest frequency factor vs axial wavelength factor: full problem, $n = 4$.

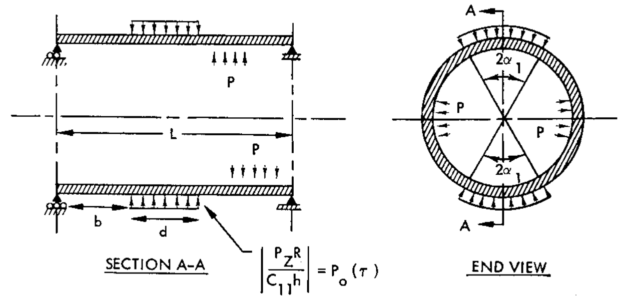

Fig. 8 Illustration of applied load, spatial distribution.

pair of axial and circumferential wave numbers, m and n. In this instance, another superscript might be added to the solution, which will now be written in the form

$$\tilde{u}(\xi,\varphi,\tau) = \sum_m\sum_n \cos m\pi\xi \cos n\varphi \sum_{r=1}^{r=3} A_{mn}{}^{(r)} q_{mn}{}^{(r)}(\tau)$$

$$\tilde{v}(\xi,\varphi,\tau) = \sum_m\sum_n \sin m\pi\xi \sin n\varphi \sum_{r=1}^{3} B_{mn}{}^{(r)} q_{mn}{}^{(r)}(\tau) \quad (31)$$

$$\tilde{w}(\xi,\varphi m\tau) = \sum_m\sum_n \sin m\pi\xi \cos n\varphi \sum_{r=1}^{3} C_{mn}{}^{(r)} q_{mn}{}^{(r)}(\tau)$$

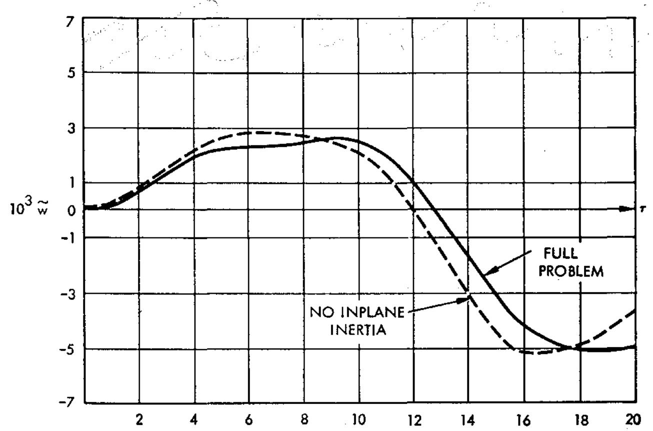

Fig. 9 Radial displacement vs time: isotropic.

The remaining equations for the evaluation of the time coordinates are also suitably modified.

The solution obtained when the tangential inertia terms are deleted is, of course, much simpler, for then there is only one frequency for each pair of wave numbers. A slight drawback arises in this case, though, in that the normal mode solution in the absence of the inplane inertia terms will not accommodate surface tractions in either the axial or the circumferential direction.

The particular loading case treated here is illustrated in Fig. 8. Spatially the load represents a portion of a ring loading, while the temporal variation is taken as a step function applied at $t = \tau = 0$. The results are presented in Table 4 and Figs. 9 to 12 for the following numerical values:

$$p_z R/C_{11}h = 0.001, \; t_{y0} = 0.001, \; b/L = d/L = 0.30 \quad (32)$$

$$\alpha_1 = \pi/4, \; \xi = 0.50, \; \varphi = 0$$

$$\text{Isotropic: } \alpha = 0.30, \beta = 1.00, \gamma = 0.35$$

$$\text{Orthotropic: } \alpha = 0.30, \beta = 0.20, \gamma = 0.30$$

Thus, the applied pressure is equal in magnitude to the internal presssure, and the location at which the response is observed is near the center of the loading ring.

Table 4 presents peak values of the stresses and displacements, computed with and without the in-plane inertia terms. The agreement appears to be quite good, which would imply that the simpler approach without the in-plane inertia terms, should be quite adequate for design analyses. It is also interesting to note that if the stiffness ratio β is reduced, the stress levels are not greatly affected, whereas the radial displacement is greatly increased.

Figures 9–12 further indicate how well the two analyses agree with each other. In all the plots, the results of the analyses without the in-plane inertia terms are almost identical to the analysis of the complete problem.

It is also interesting to note in these figures that the response in the orthotropic case, where the circumferential stiffness is reduced, is much smoother and much more in line with that which would be expected from a linear analysis. In Fig. 12, the radial displacement response for three values of the internal pressurization for an isotropic shell is plotted. It is apparent that as the internal pressure increases, with the

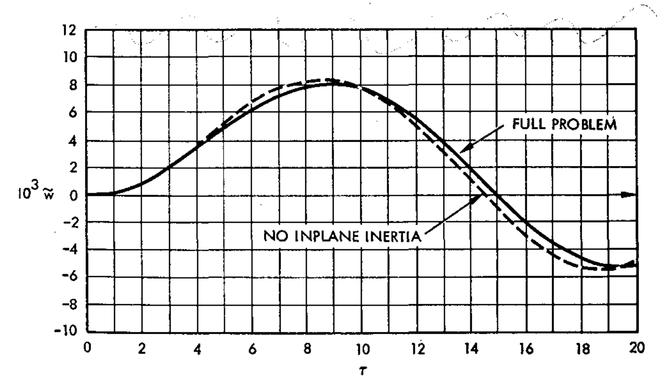

Fig. 10 Radial displacement vs time: orthotropic.

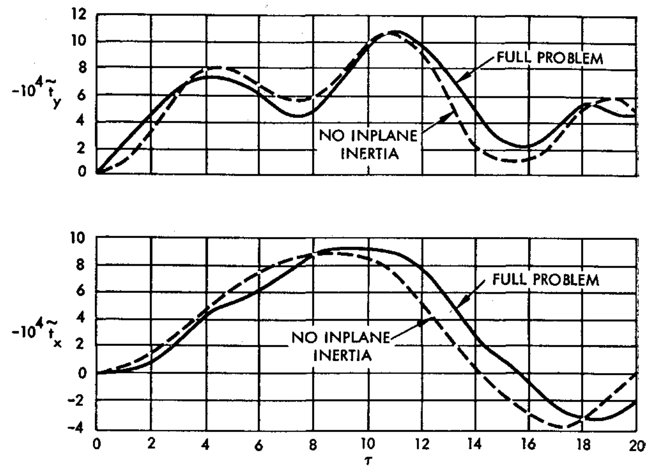

Fig. 11 Stresses vs time: isotropic.

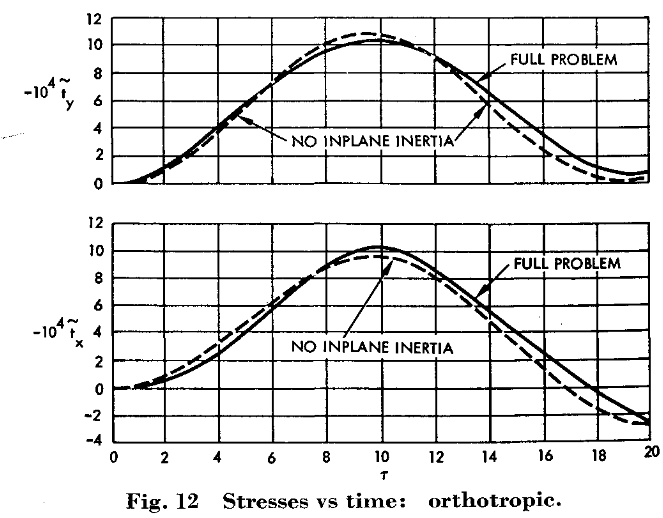

Fig. 12 Stresses vs time: orthotropic.

Table 4 Peak displacements and stresses for abrupt ring load (t's at peak enclosed in parentheses)

	Isotropic		Orthotropic	
	with in-plane inertia	no in-plane inertia	with in-plane inertia	no in-plane inertia
w	0.005216 (19)	0.005200 (17)	0.008046 (19)	0.008224 (9)
t_x	0.000910 (11)	0.000887 (19)	0.001038 (10)	0.000947 (10)
t_y	0.001081 (11)	0.001056 (11)	0.001018 (10)	0.001084 (9)

applied pressure held constant, the magnitude and form of the response change considerably. The magnitude decreases, which is readily explained by the increased stiffening of the shell due to the increased pressurization. The form of the response curve becomes smoother, and more closely resembles the periodic type of response that one might expect.

It is not difficult to conject that the responses illustrated in Figs. 9, 11, and 13 are in fact due to the nonlinearity of the system which is inherent even after the linearization procedure. Reissner[4] has pointed out that for an isotropic shell, the nonlinear effects are negligible only if (in present notation)

$$t_{y0} \gg (1 - \nu^2)\tilde{w}$$

This is confirmed by Fig. 13. In the orthotropic case, the relaxation of the circumferential stiffness apparently changes the hoop stresses sufficiently so that a new analysis would be necessary in order to predict appropriate bounds for the linearized theory.

Finally it should be noted that although the theory that retained only the radial inertia was in close agreement with the full theory, the convergence of the double sums of the full problem was faster, from the standpoint of the number of terms involved. However, the running machine time was roughly the same, because the individual terms of the full problem require more calculation.

Conclusions

The primary conclusions may be stated as follows. 1) For small circumferential wave numbers and axial wavelength factors, the free vibration frequencies are as sensitive to changes in the elastic constants as they are to changes in the internal pressure. For large λ, n, the elastic constants have little effect. 2) The deletion of in-plane inertia terms still allowed, in most cases, a very good approximation to the frequency of the radial mode, except for very small values of λ and n. 3) The solution of the forced vibration problem without the tangential inertia terms is in close agreement with the solution obtained to the full problem. 4) In the forced motion problem, the effects of pressurization and orthotropy are quite important. In fact, the justification for using a linearized analysis is quite dependent on the values assumed for elastic constants and the range of the internal pressure in the shell.

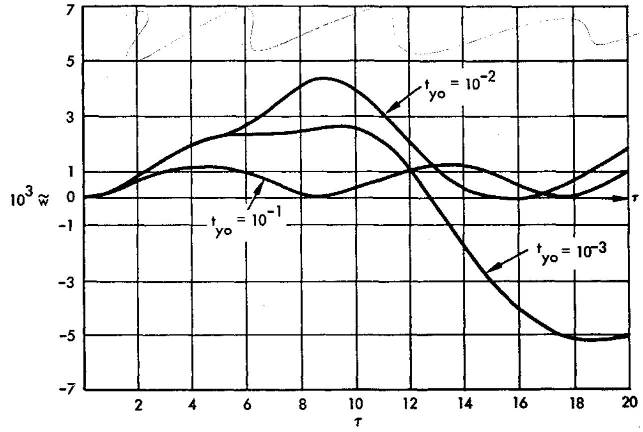

Fig. 13 Radial displacement vs time: full problem.

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